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The reduced length of a polynomial with complex or real coefficients

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Let for a polynomial $P \in \mathbb{C}[x]$, $P(x) = \sum_{i=0}^d a_i x^{d-i} = a_0 \prod_{i=1}^d (x - \alpha_i)$, $P^*(x) = \sum_{i=0}^d a_i x^i$, $L(P) = \sum_{i=0}^d |a_i|$, $M(P) = |a_0| \prod_{i=1}^d \max\{1, |\alpha_i|\}$ and let $\mathbb{C}[x]^1$, $\mathbb{R}[x]^1$ denote the set of monic polynomials over \mathbb{C} or \mathbb{R} , respectively.

$L(P)$ is called the length of P . Following A. Dubickas [1] we consider $l(P)$, the reduced length of P defined by the formula

$$l(P) = \inf_{G \in \mathbb{C}[x]^1} L(PG),$$

which for $P \in \mathbb{R}[x]$ reduces to

$$(1) \quad l(P) = \inf_{G \in \mathbb{R}[x]^1} L(PG).$$

Actually Dubickas considered only the case $P \in \mathbb{R}[x]$ and called the reduced length of P the quantity $\min\{l(P), l(P^*)\}$. For $P \in \mathbb{R}[x]$ some of the following results of [6] are due to him.

Proposition 1. Suppose that $\omega, \eta, \psi \in \mathbb{C}$, $|\omega| \geq 1$, $|\eta| < 1$, then for every $Q \in \mathbb{C}[x]$

- (i) $l(\psi Q) = |\psi| l(Q)$,
- (ii) $l(x + \omega) = 1 + |\omega|$,
- (iii) if $T(x) = Q(x)(x - \eta)$, then $l(T) = l(Q)$,
- (iv) $l(\overline{Q}) = l(Q)$, where \overline{Q} denotes the complex conjugate of Q .

Proposition 2. *For all P, Q in $\mathbb{C}[x]^1$, all $\eta \in \mathbb{C}$ with $|\eta| = 1$ and all positive integers k*

- (i) $\max\{l(P), l(Q)\} \leq l(PQ) \leq l(P)l(Q),$
- (ii) $M(P) \leq l(P),$
- (iii) $l(P(\eta x)) = l(P(x)),$
- (iv) $l(P(x^k)) = l(P(x)).$

The main problem consists in finding an algorithm of computing $l(P)$ for a given P . An apparently similar problem in which P and G in formula (1) are restricted to polynomials with integer coefficients has been considered in [2] and [3], however the restriction makes a great difference. Coming back to our problem Proposition 1 (iii) shows that it is enough to consider P with no zeros inside the unit circle. The case of zeros on the unit circle is treated in the following two theorems.

Theorem 1. *Let $P \in \mathbb{C}[x], Q \in \mathbb{C}[x]^1$ and Q have all zeros on the unit circle. Then for all $m \in \mathbb{N}$*

$$l(PQ^m) = l(PQ).$$

Theorem 2. *If $P \in \mathbb{C}[x]^1 \setminus \mathbb{C}$ has all zeros on the unit circle, then $l(P) = 2$ with $l(P)$ attained, if all zeros are roots of unity and simple ($l(P)$ is attained means that $l(P) = L(Q)$, where $Q/P \in \mathbb{C}[x]^1$).*

Proofs for $P \in \mathbb{R}[x]$ are given in [4], proofs for $P \in \mathbb{C}[x]$ are essentially the same. We have further (see [6]).

Theorem 3. *Let $P = P_0P_1$, where $P_0 \in \mathbb{C}[x]$, $P_1 \in \mathbb{C}[x]^1$, $L(P_0) \leq 2|P_0(0)|$. Then*

$$l(P) \geq L(P_0) + (2|P_0(0)| - L(P_0))(l(P_1) - 1).$$

Corollary 1. *If $P \in \mathbb{C}[x]$ and $L(P) \leq 2|P(0)|$, then*

$$l(P) = L(P).$$

Conversely, if $l(P) = L(P)$ and all coefficients of P are real and positive, then $L(P) \leq 2P(0)$.

Corollary 2. *If $P(x) = (x - \alpha)(x - \beta)$, where $|\alpha| \geq |\beta| \geq 1$, then*

$$l(P) \geq 1 + |\alpha| - |\beta| + |\alpha\beta|$$

with equality if $\alpha/\beta \in \mathbb{R}$ and either $\alpha/\beta < 0$ or $|\beta| = 1$.

Corollary 3. *Let $P = P_0P_1$, where $P_\nu \in \mathbb{C}[x]$ ($\nu = 0, 1$), $\deg P_1 \geq 1$ and all zeros z of P_ν satisfy $|z| > 1$ for $\nu = 0$, $|z| = 1$ for $\nu = 1$. If*

$$(2) \quad l(P_0) = L(P_0),$$

then

$$(3) \quad l(P) \geq 2M(P).$$

It remains a problem, whether (3) holds without the assumption (2). The following results of [6] point towards an affirmative answer.

Theorem 4. *If $P \in \mathbb{C}[x] \setminus \{0\}$ has a zero z with $|z| = 1$, then*

$$L(P) > \sqrt{2}M(P), \quad l(P) \geq \sqrt{2}M(P).$$

Theorem 5. *If $P(x) = (x - \alpha)(x - \beta)(x - 1)$, where α, β are real and at least one of them is positive, then (3) holds.*

The question of validity of (3) for all polynomials P on \mathbb{C} is equivalent to the following

Problem 1. *Is it true that for all polynomials P in $\mathbb{C}[x]$ with a zero on the unit circle $L(P) \geq 2M(P)$?*

The following theorems like Theorem 5 concern P in $\mathbb{R}[x]$.

Theorem 6 ([4], Theorem 1). *If $P \in \mathbb{R}[x]^1$ is of degree d with $P(0) \neq 0$, then $l(P) = \inf_{Q \in S_d(P)} L(Q)$, where $S_d(P)$ is the set of all polynomials in $\mathbb{R}[x]^1$ divisible by P with $Q(0) \neq 0$ and with at most $d+1$ non-zero coefficients, all belonging to the field $K(P)$, generated by the coefficients of P .*

Theorem 7 ([4], Theorem 2). *If $P \in \mathbb{R}[x]$ has all zeros outside the unit circle, then $l(P)$ is attained and effectively computable, moreover $l(P) \in K(P)$.*

Theorem 8 ([5], Theorem 1). *Let $P(x) = \prod_{i=1}^3 (x - \alpha_i)$, where α_i distinct, $|\alpha_1| \geq |\alpha_2| > |\alpha_3| = 1$. Then $l(P)$ is effectively computable.*

Theorem 9 ([5], Theorem 2). *Let $P(x) = (x - \alpha)(x^2 - \varepsilon)$, where $|\alpha| > 1$, $\varepsilon = \pm 1$. Then*

$$l(P) = 2(|\alpha| + 1 - |\alpha|^{-1}).$$

The following problem remains open

Problem 2. How to compute $l(2x^3 + 3x^2 + 4)$?

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